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A SEMI-MARKOV MODEL FOR MILITARY OFFICER PERSONNEL SYSTEMS

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THESIS

A SEMI-MARKOV MODEL FOR MILITARY OFFICER PERSONNEL SYSTEMS

by

Randall Walter Austin

Thesis Advisor: K. T. Marshall

September 1971

Approved for public release; distribution unlimited.

A Semi-Markov Model for Military Officer Personnel Systems

by

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Major, United States Marine Corps
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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 1971

ABSTRACT

The military officer personnel system is modeled as a semi-Markov process. The probability distributions describing state transitions are developed in terms of policy variables. Effects of changes in promotion policy and force size are studied for both a closed, steady state system and an open, transient situation. Moments of the joint distribution of officers in each rank are presented for all cases. A method for approximation of transient results from known distributions is formulated.

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I. INTRODUCTION

Personnel systems composed of a formal rank or grade structure that utilize a promotion procedure to move persons through or out of the system are commonplace in modern large organizations. In particular, the Officer Corps of each of the military services provides a classic example of this type.

Military officer promotions are controlled by federal laws such as the Officer Personnel Act of 1947 and the Officer Grade Limitation Act of 1954. Additional constraints may be imposed by service Secretaries, frequently acting as approving authority for the respective service's internally generated promotion plan.

Statutory ceilings, often further reduced by service policies, exist on the maximum number of officers holding a particular rank. Promotion boards meet annually to select officers for promotion to the next higher rank. Total numbers to be promoted are established in accordance with previous and projected attrition. The size of the promotion zone - those officers to be considered for selection - is based on a promotion opportunity parameter, e.g., 500 promotions might be obtained by a 50% opportunity from a 1000 man zone, or by a 80% opportunity from a zone of 625.

Time in grade - the number of years an officer has spent in his current rank - is a secondary consideration in fixing the size of the zone. That is, the maximum speed

with which an individual may move through the system is a function of the size of the zone each year.[1]

Thus, there exist three policy variables, (1) total number in each rank, (2) promotion opportunity to next rank, and (3) time in grade, which control the functioning of the promotion system.

The purpose of this paper is to present a mathematical model of the flow of personnel through this type multi-grade system and to study both long range effects of a given promotion policy and short term effects of a change in policy. The paper is written in four parts. In Part II we formulate a semi-Markov process model and define its variables and properties. Part III contains steady state results, and in Part IV we discuss transient results and derive a method to approximate them.

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[1] Accelerated promotions "below the zone" do occur. They account for less than 5% of the total promoted but will be included in all probability distributions discussed in this paper.

II. MODEL FORMULATION

Personnel flow models in which movement from state-to-state in successive time periods is assumed to follow a Markov chain have received considerable attention in the literature (e.g., Bartholomew [1], Gani [3]). An extension of a Markov chain model which accounts for some of the non-Markov properties of personnel systems is to model such a system as a semi-Markov process (S.M.P.), i.e., a process whose sequence of states forms an embedded Markov chain at the transition points, where the system changes state, and in which the lengths of time between these transitions are random variables (see Pyke [6]).

Let us first formulate an "open model." That is, if state j, j = 1,2,...,6, is the rank of an active military officer and state 7 is an absorbing state containing those officers who are retired, resigned, deceased or otherwise "out of the service," then ranks 1 through 6 are a set of transient states that form the "system." Persons leaving this set of states must be replaced by new recruits into the system if it is not to empty over time. Figure 1 illustrates movement in an open model.

By controlling the input flow of new personnel we can control the total number in the system. If that control is such that in any time period the number of departures is equal to the number of arrivals, the number in the system remains constant. Since in this paper we begin by studying

a system of fixed size (in Part IV this assumption is relaxed) we can "close" (see Cinlar [2]) the model by artificially closing the feedback loop shown as a dotted line in Figure 1.

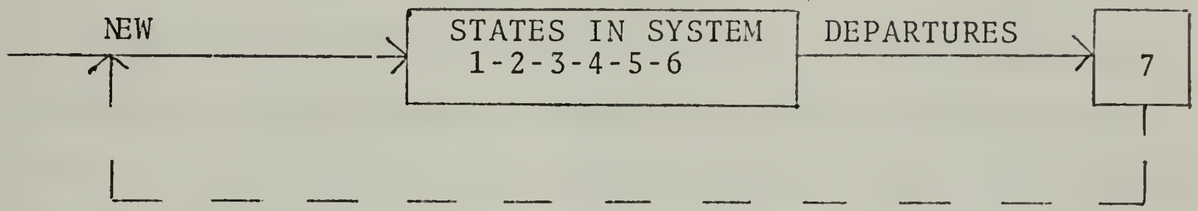

Figure 1. Personnel Movement in Open Model.

Now consider a system with 7 states, where state 7 is artificial and represents the absorbing state for those leaving active service and from which a new recruit comes to enter the active states and replace the departing officer (although he does not necessarily enter the state from which the departure occurred).

As we shall see, the times taken to move from state to state in the model play an important role, and for states 1 through 6 these times are dependent on policy decisions. By fixing the time to move from the artificial state 7 back to the active states at zero (instantaneous replacement), we can remove state 7 from the system entirely and close the loop in Figure 1 without it. Thus we are lef with a

closed system of 6 recurrent, non-null states in our semi-Markov model.

It is a system of this type which will be considered in this paper. We shall allow inputs to the system to occur only at rank 1. In Part IV we evaluate increases of overall size by new cohort inputs to the fixed size.

We now pursue the system containing states 1, 2, 3, 4, 5, 6, the officer ranks from 01 to 06.[2] As time passes an individual officer moves through the system from state to state. A transition occurs when he moves to the next higher state via promotion or when he leaves the service, in which case he returns to state 1 and is "replaced" as previously described. We now assume that the sequence consisting of points immediately following transition points forms a Markov chain. It is significant to note here that the validity of this Markovian assumption is established by the special structure of the military promotion system - an officer either moves up to the next rank or he leaves the system. Thus knowledge of only the present rank provides complete information concerning the probabilities for the next transition.

Times between transitions are random variables that depend only on the current state and the state to which the

---

[2] Rank/Grade	USA-USAF-USMC TITLE	USN TITLE
01	Second Lieutenant	Ensign
02	First Lieutenant	Lieutenant j.g.
03	Captain	Lieutenant
04	Major	Lieutenant Commander
05	Lieutenant Colonel	Commander
06	Colonel	Captain

transition will be made. Implicit here is the assumption that accelerated or delayed promotions do not affect later behavior.

Formally then, consider the state space $\{1, 2, 3, 4, 5, 6\}$[3], the first six officer ranks. Let $\{Z_k(t) = i\}$ be the event officer k is in rank i after a time t in service. Then the stochastic process $\{Z_k(t), t \geq 0\}$ is a semi-Markov process (S.M.P.). The stochastic process that records the number of times the S.M.P. has entered each of the system states is called a Markov Renewal Process (M.R.P.), as named by Pyke [6,7]. The S.M.P. and the associated M.R.P. are usually discussed together in the literature. However, the special structure of a personnel system is such that the following discussion will be limited to the S.M.P.

We shall assume that each officer in the system follows an independent path from the same S.M.P.[4] If we let $X_j(t)$ be the number of officers in rank j at a time t, then $\{X(t), t \geq 0\}$ where $X(t) = (X_1(t), X_2(t), \ldots, X_6(t))^T$, is a multi-particle semi-Markov process generated by the S.M.P.

We can now define

$$\begin{aligned} Q_{ij}(t) &= P\left[\begin{array}{l|l} \text{process next makes a transi-} & \text{process has} \\ \text{tion into j } \underline{\text{and}} \text{ this occurs} & \text{just entered} \\ \text{in a time} \leq \text{t} & \text{i} \end{array}\right] \\ &= P\left[\begin{array}{l|l} \text{officer k next goes to rank} & \text{he has just} \\ \text{j } \underline{\text{and}} \text{ this occurs in a time} & \text{attained rank} \\ \leq \text{t} & \text{i} \end{array}\right] \end{aligned} \qquad (1)$$

---

[3] The state space could, of course, be described in general. It is limited to the specific case to be discussed to simplify subsequent notation.

[4] See Heyman [4] definition of a Multi-particle SMP.

$$\begin{aligned}
P_{ij}(t) &= P\left[\begin{array}{l}\text{state at time } t \text{ is } j\end{array} \;\middle|\; \begin{array}{l}\text{process}\\ \text{started in}\\ \text{state } i \text{ at}\\ \text{time } 0\end{array}\right]\\
&= P\left[\begin{array}{l}\text{officer's rank at time}\\ t \text{ is } j\end{array} \;\middle|\; \begin{array}{l}\text{officer}\\ \text{started in}\\ \text{rank } i \text{ at}\\ \text{time } 0\end{array}\right]\\
&= P\,[Z_k(t) = j \mid Z_k(0) = i].
\end{aligned} \tag{2}$$

$$\begin{aligned}
F_{ij}(t) &= P\left[\begin{array}{l}\text{a transition will occur}\\ \text{in an amount of time} \le t\end{array} \;\middle|\; \begin{array}{l}\text{process has}\\ \text{just entered}\\ i \text{ and will}\\ \text{next enter } j\end{array}\right]\\
&= P\left[\begin{array}{l}\text{officer's rank will}\\ \text{change in an amount of}\\ \text{time} \le t\end{array} \;\middle|\; \begin{array}{l}\text{officer has}\\ \text{just attained}\\ \text{rank } i \text{ and}\\ \text{will next}\\ \text{hold rank } j\end{array}\right]
\end{aligned} \tag{3}$$

$$\begin{aligned}
H_i(t) &= P\left[\begin{array}{l}\text{next transition occurs}\\ \text{in an amount of time}\\ \le t\end{array} \;\middle|\; \begin{array}{l}\text{process has}\\ \text{just entered}\\ i\end{array}\right]\\
&= P\left[\begin{array}{l}\text{time officer spends in}\\ \text{rank } i \le t\end{array} \;\middle|\; \begin{array}{l}\text{he has just}\\ \text{attained}\\ \text{rank } i\end{array}\right]
\end{aligned} \tag{4}$$

$$G_{ij}(t) = P\left[\begin{array}{l}\text{first transition into}\\ \text{state } j \text{ occurs in a time}\\ \le t\end{array} \;\middle|\; \begin{array}{l}\text{process}\\ \text{starts in } i\end{array}\right] \tag{5}$$

It should be noted that $G_{ij}(t)$ does not have a meaningful explanation in terms of the functioning of the officer system. However, its usefulness in the derivation of further results merits its inclusion in the development.

Since $Q_{ij}(\infty)$ is the probability that the next transition is into state j given the process has just entered i, we define:

$$Q_{ij}(\infty) = p_{ij}$$

so that p_{ij} is the fraction of those who reach rank i who ever attain rank j.

Denoting the matrix of $Q_{ij}(t)$ by $\mathcal{Q}(t)$ [we will henceforth use the notation $\mathcal{Q}(t) = (Q_{ij}(t))$] we write

$$\mathcal{Q}(\infty) = \begin{bmatrix} q_{11} & p_{12} & 0 & 0 & 0 & 0 \\ q_{21} & 0 & p_{23} & 0 & 0 & 0 \\ q_{31} & 0 & 0 & p_{34} & 0 & 0 \\ q_{41} & 0 & 0 & 0 & p_{45} & 0 \\ q_{51} & 0 & 0 & 0 & 0 & p_{56} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (6)$$

where $q_{i1} = 1 - p_{i\,i+1}$ is the fraction of those who reach rank i but are never promoted to rank i+1 and are replaced by new officers in rank 1.

It should be re-emphasized that this matrix defines a closed system - a system in which the number of officers N remains fixed with no extra entering or leaving the system over time. Thus, mathematically we have a certain fraction leaving each state and returning to state 1 with the interpretation of instantaneous replacement by a new officer.

Recalling that $Q_{ij}(t)$ is the joint probability of the next transition from i to j and in a time less than t we have

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Denoting the matrix of $Q_{ij}(t)$ by $\mathcal{Q}(t)$ [we will henceforth use the notation $\mathcal{Q}(t) = (Q_{ij}(t))$] we write

$$\mathcal{Q}(\infty) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} q_{11} & p_{12} & 0 & 0 & 0 & 0 \\ q_{21} & 0 & p_{23} & 0 & 0 & 0 \\ q_{31} & 0 & 0 & p_{34} & 0 & 0 \\ q_{41} & 0 & 0 & 0 & p_{45} & 0 \\ q_{51} & 0 & 0 & 0 & 0 & p_{56} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (6)$$

where $q_{i1} = 1 - p_{ii+1}$ is the fraction of those who reach rank i but are never promoted to rank i+1 and are replaced by new officers in rank 1.

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Recalling that $Q_{ij}(t)$ is the joint probability of the next transition from i to j and in a time less than t we have

$$F_{ij}(t) = \frac{Q_{ij}(t)}{Q_{ij}(\infty)} = \frac{Q_{ij}(t)}{p_{ij}} ,$$

and thus

$$H_i(t) = \sum_{j=1}^{6} p_{ij} \, F_{ij}(t) = \sum_{j=1}^{6} Q_{ij}(t).$$

It can be shown (Pyke [6]) that

$$P_{ij}(t) = \delta_{ij} \, [\, 1 - H_i(t) \,] + \sum_{k=1}^{6} \int_{0}^{t} P_{kj}(t-x) dQ_{ik}(x)$$

where δ_{ij} is the Kroneker delta.

If we let $\overline{H}_i(t) = 1 - H_i(t)$ and $Q_{ij}(t) * P_{kj}(t)$ indicate the convolution we obtain

$$P_{ij}(t) = \delta_{ij} \, \overline{H}_i(t) + \sum_{k=1}^{6} Q_{ij} * P_{kj}(t) \quad i,j = 1,2,\ldots,6.$$

The matrix form of these equations is

$$P(t) = \overline{H}_D(t) + Q * P(t) \tag{7}$$

where $\overline{H}_D(t)$ is a diagonal matrix of $\overline{H}_i(t)$, and $Q(t) * P(t)$ is the matrix convolution.[5]

A similar relationship between $G_{ij}(t)$ and $Q_{ij}(t)$ can be defined as

$$G_{ij}(t) = \sum_{k=1}^{6} \int_{0}^{t} G_{kj}(t-x) dQ_{ik}(x) + \int_{0}^{t} 1 - G_{jj}(t-x) dQ_{ij}(x) \tag{8}$$

with the matrix form

---

[5] A convolution of matrix valued functions is similar to matrix multiplication except the product of elements is replaced by the convolution of elements.

$$G(t) = Q * G(t) + Q(t) - Q * G_D(t). \qquad (9)$$

In terms of the policy variables of Part I we can now define (Ross [8]) the average amount of time spent in rank i as

$$\mu_i = \int_0^\infty t \, d\, H_i(t) = \int_0^\infty \overline{H}_i(t),$$

and the average time spent in rank i given the next transition is to rank j

$$\eta_{ij} = \int_0^t t \, dF_{ij}(t) = \int_0^\infty \left(1 - \frac{Q_{ij}(t)}{p_{ij}} \right) dt.$$

Finally, the mean first passage time from i to j is

$$\mu_{ij} = \int_0^\infty t \, d\, G_{ij}(t) \qquad \text{for } G_{ij}(\infty) = 1.$$

Although the concept of first passage is of little direct use in this model, it should be noted that $G_{11}(\cdot)$ is the distribution of lifetime in the system. For the most part, however, $G(t)$ will be used to derive other results which are in terms of the policy variables.

In the officer promotion system we can consider $Q(t)$ (and thus $\overline{H}_D(t)$, μ_i and η_{ij}) as known since the $Q_{ij}(t)$ are determined by promotion policies and officer behavior patterns. Parts III and IV will discuss the unknown $P(t)$, whose first row is of primary importance. $NP_{ij}(t)$, j = 1,2,...,6, gives the expected number of the N officers who are in rank j at time t. For large t, the P_{ij} t) can be

interpreted as the fraction of officers expected to be in rank j. Thus, to evaluate the long term effect of a given policy, the limiting form of $P(t)$ is required and its determination directly from (7) is formidable. The introduction of the distribution $G(\cdot)$ and in particular (9) will simplify the task.

Measurements of the limiting form of $P(t)$ could, of course, be done directly via a cohort type model (e.g., Marshall, [5]), but these direct measurements do not give any information as to how changes in policy affect the $P_{ij}(t)$'s.

III. STEADY STATE RESULTS

Let us first examine the output of this model after a long period of time; that is, in the steady state.

In the steady state, the limiting form of $P(t)$ contains the distribution of the N officers over the 6 states. We shall see that its solution is indirectly in terms of policy variables and will illustrate how the effects of policy changes can be measured in terms of the parameters of the model.

Defining the Laplace Stieltjes Transform (L.S.T.) of the function F as

$$\hat{F}(s) = \int_0^\infty e^{-sx}\, d\, F(x) \qquad s \geq 0$$

we solve simultaneously equations (7) and (9).

First, taking transforms of (7) we obtain

$$\hat{P}(s) = \hat{\hat{H}}_D(s) + \hat{Q}(s)\, \hat{P}(s), \tag{10}$$

which can also be written

$$[I-\hat{Q}(s)]^{-1} = \hat{P}(s)\, \hat{\hat{H}}_D(s)^{-1}. \tag{11}$$

Likewise, transforms in (9) yield

$$\hat{G}(s) = \hat{Q}(s) - \hat{Q}(s)\, G_D(s) + \hat{Q}(s)\, \hat{G}(s) \tag{12}$$

which can be simplified to

$$[I-\hat{Q}(s)]^{-1} = I + \hat{G}(s)\, (I-G_D(s))^{-1}. \tag{13}$$

Equating (11) and (13) we have the result

$$\hat{P}(s) = \hat{\hat{H}}_D(s) + \hat{G}(s)\, \hat{A}_D(s) \tag{14}$$

where $\hat{A}_D(s)$ is a diagonal matrix with elements

$$\frac{\hat{H}_i(s)}{\hat{\bar{G}}_{ii}(s)} .$$

To analyze the steady state results we now find $\lim_{t\to\infty} P(t)$ or equivalently the $\lim_{s\to 0} \hat{P}(s)$. Thus,

$$\lim_{s\to 0} \hat{P}(s) = \lim_{s\to 0} \hat{\bar{H}}_D(s) + \lim_{s\to 0} \hat{G}(s) \lim_{s\to 0} \hat{A}_D(s) = 0$$

$$+ \mathit{1} \lim_{s\to 0} \begin{bmatrix} \dfrac{H_1(s)}{\bar{G}_{11}(s)} & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & \dfrac{\hat{\bar{H}}_6(s)}{\hat{\bar{G}}_{66}(s)} \end{bmatrix}$$

where $\mathit{1}$ is a 6x6 matrix of 1's.

Now, using L'Hospital's rule we see that

$$\lim_{s\to 0} \frac{\hat{\bar{H}}_i(s)}{\hat{\bar{G}}_{ii}(s)} = \lim_{s\to 0} \frac{\hat{\bar{H}}_i'(s)}{\hat{\bar{G}}_{ii}'(s)} = \frac{\mu_i}{\mu_{ii}} ,$$

so that finally we can say

$$\lim_{t\to\infty} \hat{P}(t) = \lim_{s\to 0} \hat{P}(s) = \begin{bmatrix} \dfrac{\mu_1}{\mu_{11}} & \dfrac{\mu_2}{\mu_{22}} & \cdots\cdots & \dfrac{\mu_6}{\mu_{66}} \\ \dfrac{\mu_1}{\mu_{11}} & \dfrac{\mu_2}{\mu_{22}} & \cdots\cdots & \dfrac{\mu_6}{\mu_{66}} \\ \vdots & & & \\ \dfrac{\mu_1}{\mu_{11}} & \dfrac{\mu_2}{\mu_{22}} & \cdots\cdots & \dfrac{\mu_6}{\mu_{66}} \end{bmatrix} . \qquad (15)$$

We will now define (see also Ross [8]) the terms in each row of this matrix of limiting probabilities as

$$P_j = \lim_{t\to\infty} P_{ij}(t) = \frac{\mu_j}{\mu_{jj}}\ , \quad j = 1,2,\ldots,6, \qquad (16)$$

that is, in the steady state the fraction of officers in rank j equals the ratio of expected time in rank j to expected recurrence time from rank j back to j.

However, since it was previously noted that, with the exception of μ_{11}, the μ_{jj} have no recognizable meaning within a personnel system structure it is desirable to develop an expression for $P_{1j}(\infty) = P_j$ which is in terms of policy variables.

Consider the embedded Markov chain and the long run proportion of transitions into each rank j; that is, the solution vector $\{\Pi_j,\ j = 1,2,\ldots,6\}$ to

$$\Pi = \Pi \quad Q(\infty).$$

Using $Q(\infty)$ as structured in (6) we arrive at

$$\pi_1 = \frac{1}{1 + \sum_{j=2}^{6} \prod_{k=1}^{j-1} p_{k,k+1}}$$

$$\pi_\ell = \frac{\prod_{k=1}^{\ell-1} p_{k,k+1}}{1 + \sum_{j=2}^{6} \prod_{k=1}^{j-1} p_{k,k+1}} \qquad \ell = 2,3,\ldots 6.$$

It can also be shown[6] that

---

[6] These results on the expression of P_j in terms of π_j are due to Ross [8], pp. 104-105.

$$\mu_{ij} = \mu_i + \sum_{k \neq j} p_{ik} \mu_{kj}$$

and combining this with $P_j = \mu_j / \mu_{jj}$ we have

$$P_j = \frac{\pi_j \mu_j}{\sum_{i=1}^{6} \Pi_j \; \mu_j} \qquad j = 1,2,\ldots 6.$$

By substituting the values of Π_j derived above we obtain the key result

$$P_j = \frac{\mu_j \prod_{k=1}^{j-1} p_{k,k+1}}{\mu_1 + \sum_{i=2}^{6} \mu_i \prod_{k=1}^{i-1} p_{k,k+1}} \qquad j = 1,2,\ldots,6 \quad (17)$$

where the numerator of P_1 is defined as μ_1. Hence, equation (17) is an expression for the fraction of officers in the system that are in rank j in the steady state, in terms of mean times in grade and elements of $Q(\infty)$ - six μ_i and five p_{ij}. Thus each P_j is a function of at most eleven parameters, but can be computed without use of the detailed structure of $Q_{ij}(t)$ or $H_i(t)$.

It is interesting to note the effect of parameter changes in (17). For any j it is easy to verify that when P_{ii+1} is increased(decreased):

P_j increases(decreases) $\quad i < j$

P_j decreases(increased) $\quad i \geq j$.

To evaluate the sensitivity of P_j to changes in a particular μ_i, let us first consider the case of a change of a

specified amount. The denominator of (17)

$$\mu_1+\mu_2P_{12}+\mu_3P_{12}P_{23}+\mu_4P_{12}P_{23}P_{34}+\mu_5P_{12}P_{23}P_{34}P_{45}$$
$$+\mu_6P_{12}P_{23}P_{34}P_{45}P_{56}$$

then is affected most by a change in μ_1, since the coefficients of μ_i $i = 2,..6$ are less than unity. Similarly, P_j becomes less sensitive to this type change in μ_i as i increases from 1 to 6. However, if a percentage change in a μ_i is considered then the actual values of the μ_i and p_{ij} will determine the μ_i to which P_j is most sensitive. For a set of parameter values that approximate current policy in all military services, μ_2 is the most critical parameter for percentage change analysis.

The case where all $\mu_i=\mu$ and all $p_{ij}=p$ provides further insight into the effect of parameter changes. Here (17) becomes

$$P_j = \frac{\mu\ p^{j-1}}{\mu + \sum_{i=2}^{6} \mu\ p^{i-1}} = \frac{p^{j-1} - p^j}{1-p^6},$$

which shows that in this special case P_j is independent of μ. Figure 2 gives the plot of P_j versus j for various values of p.

Returning to the concept of the MSMP(Multiparticle) we can now consider the joint distribution of the number of officers in each rank. Since P_j is the probability of reaching state j in a single trial, and the movement of each officer

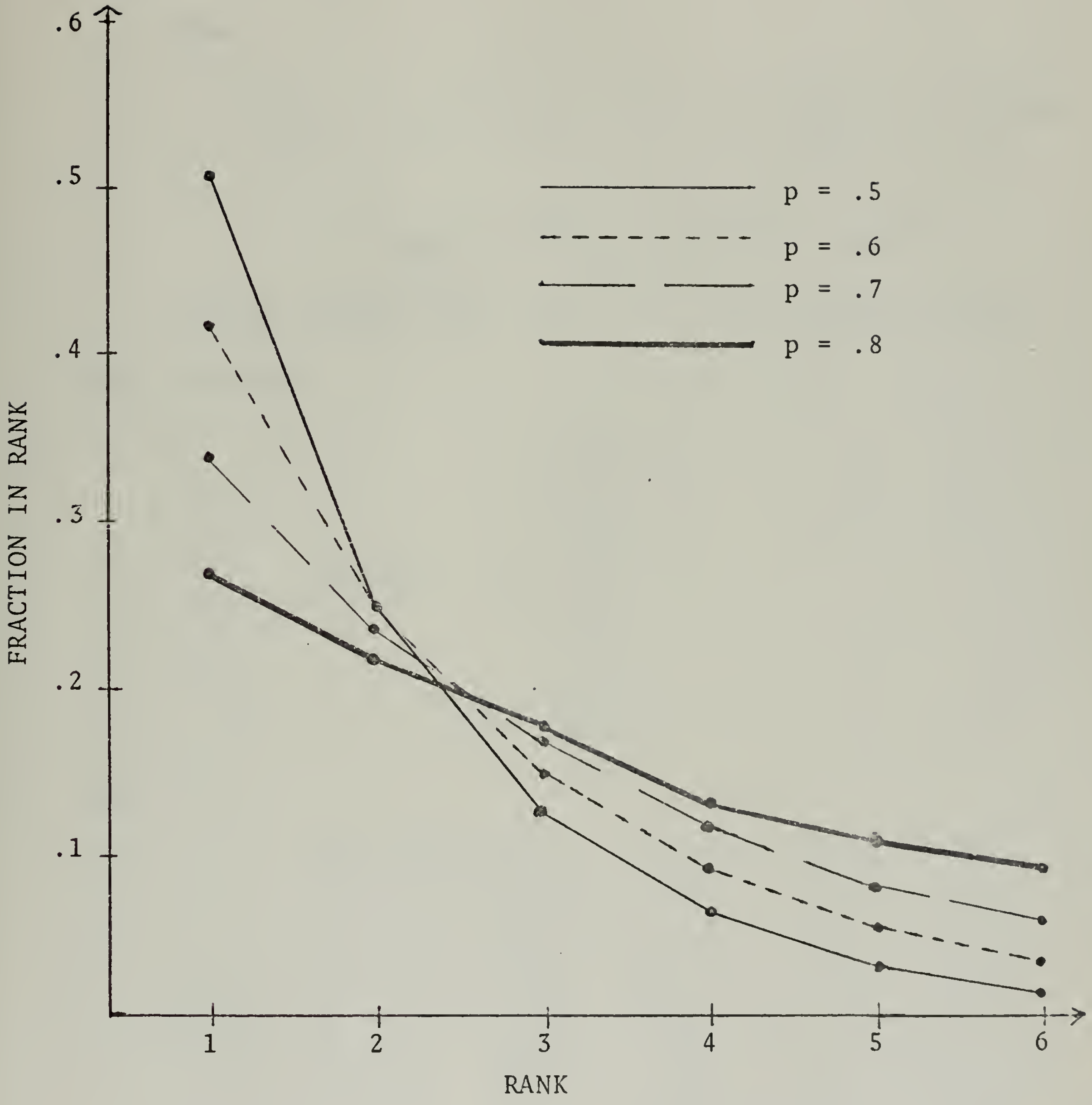

Figure 2. Distribution in Rank for $p_{ij} = p$.

is an independent path, the joint distribution is multinomial; that is, $X(\infty) = (X_1, X_2, \ldots, X_6)^T$ is distributed multinomial with parameters N and P where $P = [P_1, P_2, \ldots, P_6]^T$, a row from $P(\infty)$.

Thus,

$$P\left[X_1 = x_1,\ X_2 = x_2, \ldots X_6 = x_6 \;\middle|\; \sum_{i=1}^{6} X_i = N \right] \quad (18)$$

$$= \frac{N!}{x_1!\ x_2!\ \ldots\ x_6!} P_1^{x_1} P_2^{x_2} \ldots P_6^{x_6}.$$

For the steady state effect of a given policy we can then say that

$$E[X(\infty)] = N \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}$$

and (19)

$$V[\bar{X}(\infty)] = N \begin{bmatrix} P_1(1-P_1) & -P_1P_2 & -P_1P_3 & -P_1P_4 & -P_1P_5 & -P_1P_6 \\ -P_2P_1 & P_2(1-P_2) & -P_2P_3 & -P_2P_4 & -P_2P_5 & -P_2P_6 \\ -P_3P_1 & -P_3P_2 & P_3(1-P_3) & -P_3P_4 & -P_3P_5 & -P_3P_6 \\ -P_4P_1 & -P_4P_2 & -P_4P_3 & P_4(1-P_4) & -P_4P_5 & -P_4P_6 \\ -P_5P_1 & -P_5P_2 & -P_5P_3 & -P_5P_4 & P_5(1-P_5) & -P_5P_6 \\ -P_6P_1 & -P_6P_2 & -P_6P_3 & -P_6P_4 & -P_6P_5 & P_6(1-P_6) \end{bmatrix}.$$

Hence, X_i and X_j are not independent, but are negatively correlated such that

$$\rho(X_i,X_j) = -\frac{P_i \; P_j}{\sqrt{(1-P_i)(1-P_j)}} \; .$$

For large N the distribution of $X(\infty)$ is approximately multivariate normal. Thus the distributions of each X_i are approximately univariate normal with mean and variance given by the appropriate elements of the mean vector and covariance matrix above; e.g.,

$$\frac{X_3 - NP_3}{\sqrt{NP_3(1-P_3)}} \quad \overset{\cdot}{\sim} \quad \Phi(0,1).$$

This enables us to make probability statements about the number of officers in any rank in the steady state.

Returning to the simplified case of just two parameters, we illustrate the correlation between the numbers of officers in two different ranks. Namely, for p = .7

$$\rho(X_3,X_5) = \frac{-P_3 \; P_6}{\sqrt{(1-P_3)(1-P_6)}}$$

$$= -.021.$$

Thus, the effect of a change in the number in rank 3 on the number in rank 6 (and vice versa) is small.

It is not the purpose of this paper to describe detailed methods of finding the distributions or estimating the parameters used in this model. It must be noted, however, that the μ_i and p_{ij} are not themselves the policy variables of a personnel system as described in Part I, but rather are functions of those policy variables The p_{ij}

represent the fraction of those that enter rank i that go on to rank j. They say nothing about attrition that removes officers from the system (sends them back to rank 1) prior to their consideration for promotion. For example, consider p_{23}. Let us assume that the expiration of tours of obligated service (EOS) for a group of officers occurs during the time they are serving in rank 2. If 2/3 leave the service at EOS or before with terminal grade 2, only 1/3 are later considered for promotion to rank 3. If the promotion opportunity to rank 3 (for that 1/3) is .9, then we may approximate

$$p_{23} = (.33)(.9) = .297.$$

From the discussion in Part II on the various moments used in the model, it can be seen that

$$\mu_i = \sum_{j=1}^{6} p_{ij}\,\eta_{ij} \qquad i = 1,2,\ldots,6. \tag{20}$$

The η_{ij} define the expected time spent in rank i, given the next rank is to be j. For each i the summation in (20) contains only two terms, i.e., $p_{i1}\eta_{i1} + p_{ii}\eta_{ii+1}$. The η_{ii+1}, expected time in grade of those to be promoted to the next rank, might be considered an actual policy variable. The η_{i1} are consequences of decisions on the η_{ii+1}, promotion opportunity, and the various other forms of attrition. Assuming three years in grade 2 before promotion, 1.5 years in grade 1, three years obligated service, and $p_{12} = .30$ we may approximate

$$\mu_2 = (.70)(1.5) + (.30)\,3 = 1.95.$$

To conclude this section let us consider a more complete example of this steady state analysis using a set of parameters developed from the above thoughts and rough approximations to current military policies.

Let

$$\begin{array}{ll} \mu_1 = 1.5 & p_{12} = .95 \\ \mu_2 = 2 & p_{23} = .3 \\ \mu_3 = 5 & p_{34} = .8 \\ \mu_4 = 5 & p_{45} = .75 \\ \mu_5 = 6 & p_{56} = .6 \\ \mu_6 = 7 & \end{array}$$

Using (17) we calculate the fractions expected in each rank in the steady state. Table I presents these results, the long term effect of increasing μ_4 to six years, and the actual 1971 distribution in the U. S. Marine Corps for comparison.[7]

TABLE I. FRACTIONS IN RANK

Rank	Approx. Parameters	Increased μ_4	Actual USMC
1	.195	.188	.132
2	.247	.239	.319
3	.185	.180	.275
4	.147	.171	.159
5	.133	.128	.080
6	.093	.094	.035

From (19) and the above results it follows:

---

[7] Source of the Marine Corps data is Combined Lineal List of Officers on Active Duty in the Marine Corps. 1 January 1971, NAVMC P-1005-PD.

$$V[X(\infty)] = N \begin{bmatrix} .157 & -.048 & -.036 & -.029 & -.026 & -.018 \\ -.048 & .185 & -.046 & -.036 & -.033 & -.023 \\ -.036 & -.046 & .151 & -.027 & -.025 & -.017 \\ -.029 & -.036 & -.027 & .125 & -.018 & -.014 \\ -.026 & -.033 & -.025 & -.018 & .115 & -.012 \\ -.018 & -.023 & -.017 & -.014 & -.012 & .084 \end{bmatrix} .$$

Inspection of the covariance matrix shows $V(X_2)$ as the largest and $V(X_6)$ as the smallest. However, from Table I we note that the numbers of officers in ranks 2 and 6 are largest and smallest respectively. To provide a more valid comparison of the variation in each of the ranks, we employ the coefficients of variation:

$$CV_j = \frac{\sqrt{NP_j(1-P_j}}{NP_j} .$$

Using N = 20,000[8] these coefficients were computed using the results from the approximate parameters and are listed in Table II. This table also lists for each X_j the interval extending two standard deviations on either side of the mean in which we expect .955 of all realizations to fall.

Thus, we may conclude that when normalized by the number of people in that rank, the variation is greatest in rank 6 and decreases as j decreases to 2. This is, of course, intuitively appealing since the officers in state 6 have experienced the largest number of state transitions.

---

[8] 1 January 1971 U. S. Marine Corps total was 21,320.

TABLE II. MEASURES OF VARIATION

Rank	Coefficient of Variation	$NP_j \pm 2\sqrt{NP_j(1-P_j)}$
1	.014	3900 ± 112
2	.012	4940 ± 121
3	.015	3700 ± 109
4	.017	2940 ± 100
5	.018	2660 ± 96
6	.022	1860 ± 82

This completes the remarks on the steady state. In the following section, we shall consider the effect of an increase or decrease in the total number of officers in the system and evaluate transient results.

IV. TRANSIENT RESULTS

Realistically, a graded personnel system which undergoes even occasional policy changes never actually reaches a steady state. Thus, analysis of the transient reaction of the system to a given policy change is necessary if the effects of such a change are to be well understood.

In (2) we defined $P_{ij}(t)$ as the probability an officer's rank at t is j, given he started in i at time zero. Throughout this paper we have assumed that $Q(t)$ can be constructed from known data and can be considered as given. Even so, the direct determination of $P(t)$ appears to be a difficult task. Hence, it is desirable to develop a method to "accurately" approximate $P(t)$.

The relationship between $P(t)$ and $Q(t)$ was stated in (7) as

$$P(t) = \overline{H}_D(t) + Q * P(t),$$

and taking Laplace Stieltjes transforms and rewriting (11) we obtain

$$\hat{P}(s) = \sum_{n=0}^{\infty} \hat{Q}(s)^n \,\hat{\overline{H}}_D(s),$$

so that, after inverting,

$$P(t) = \sum_{n=0}^{\infty} Q^{(n)} * \overline{H}_D(t), \tag{21}$$

where $Q^{(n)}(t)$ is the n^{th} fold convolution of $Q(t)$ with itself and $Q^{(0)}(t) = I$.

Since the expression for $P(t)$ in (21) requires the matrix convolution of $Q(t)$ with itself for all $n = 1,2,3,\ldots$ we seek simpler expressions which give bounds or approximations to $P(t)$.

First, 1 is certainly an upper bound for $P_{ij}(t)$. If we substitute $P(t) = 1$ in the right hand side of (7) we can write:

$$P(t) \leq \overline{H}_D(t) + Q * 1 = \overline{H}_D(t) + H(t),$$

where $H(t)$ is a matrix whose i^{th} row contains elements $H_i(t)$. A second iteration, substituting the right hand side above for $P(t)$ yields

$$\begin{aligned} P(t) &\leq \overline{H}_D(t) + Q * [\overline{H}_D(t) + H(t)] \\ &\leq \overline{H}_D(t) + [Q * \overline{H}_D(t)] + [Q * H(t)], \end{aligned}$$

and by continuing this process, we have after n+1 iterations

$$P(t) \leq \sum_{k=0}^{n} Q^{(k)} * \overline{H}_D(t) + Q^{(n)} * H(t). \qquad (22)$$

As $n\to\infty$ the term $Q^{(n)} * H(t) \to 0$ so that this iterative process does converge to $P(t)$ as defined in (21).

Certainly, $P_{ij}(t) \geq 0$, and an identical iterative calculation shows that

$$P(t) \geq \sum_{k=0}^{n} Q^{(k)} * \overline{H}_D(t). \qquad (23)$$

Thus, by combining (22) and (23) we have

$$0 \leq P(t) - \sum_{k=0}^{n} Q^{(k)} * \overline{H}_D(t) \leq Q^{(n+1)} * 1. \qquad (24)$$

By choosing n large enough we can estimate $P(t)$ by

$$\sum_{k=0}^{n} Q^{(k)} * \bar{H}_D(t)$$

to any desired accuracy.

In Part II a system of constant size N was considered and no changes in total size over time were investigated. Let us now consider an increase in total size generated by increased inputs into state 1, i.e., the number of new recruits to the system exceeds the number departing from the system in a time period. This may be done on a one-time basis at a time we shall call zero, or by adding different amounts each year for several successive years, starting at time zero. Figure 3 depicts this type increase.

Let n(t) = the number added at time t which are in excess of those needed to replace departures from the system. Then N(t), the total system size at time t may be defined as

$$N(t) = N + \sum_{u=0}^{t} n(u) \quad . \tag{25}$$

This system is still "closed" and the concept of immediate replacement of departures applies for the entire N(t). If we assume the original system of N officers was in the steady state at time zero, the analysis of that group can be conducted as discussed in Part III. To derive information concerning the entire system we can superimpose the transient results of the added cohorts onto the original system results.

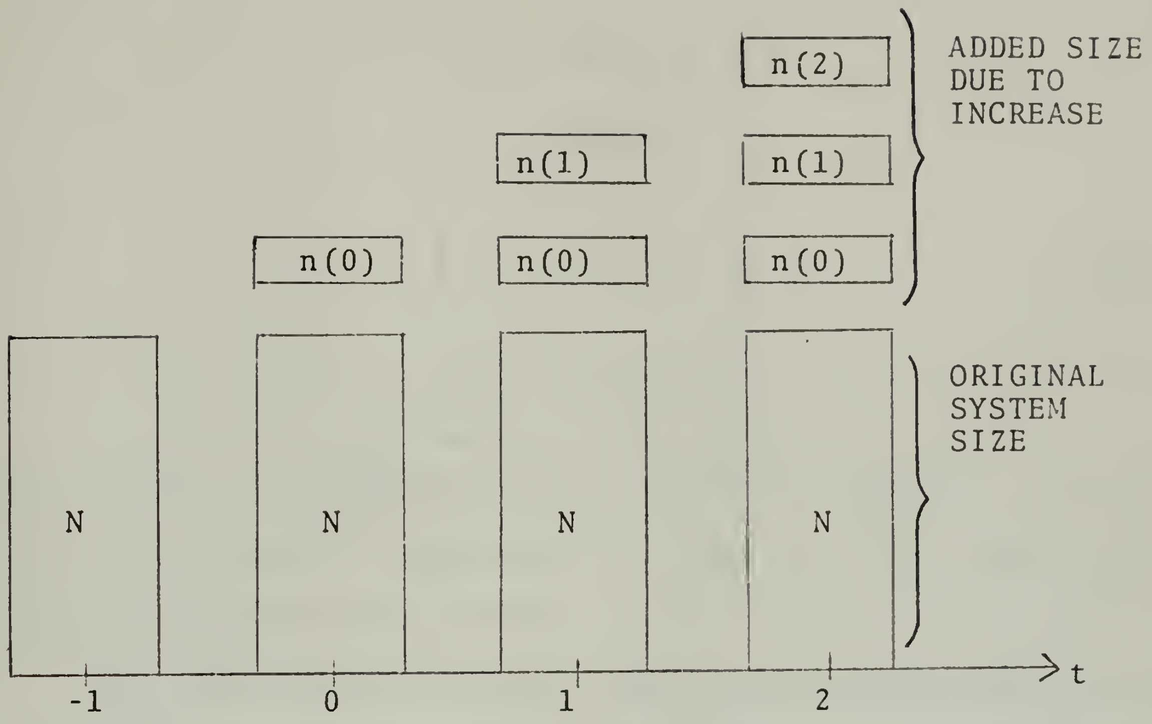

Figure 3. Size Increase.

Let $Y_j(t)$ be the number of officers from those extra cohorts added at time zero and after that are in rank j at time t. Thus

$$Y(t) = [Y_1(t), Y_2(t), \ldots, Y_6(t)]^T.$$

Similarly define $S_j(t)$ and the vector S(t) for the total system. Then

$$E[Y_j(t)] = \sum_{u=0}^{t} n(t-u)\; P_{1j}(u) \tag{26}$$

and the vector of expected numbers of officers in the total system that are in the various ranks at time t is given by

$$E[S(t)] = E[X(\infty)] + E[Y(t)]$$

$$= \begin{bmatrix} NP_1 + E[Y_1(t)] \\ NP_2 + E[Y_2(t)] \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ NP_6 + E[Y_6(t)] \end{bmatrix} . \qquad (27)$$

We see that to calculate (27), $E[Y_j(t)]$ is needed, and from (26) this requires knowledge of $P_{1j}(u)$, $u \leq t$. These $P_{1j}(t)$'s can be approximated by (24).

As discussed previously, the joint distribution of the number in each rank at time t for a particular cohort input is multinomial. By assumption, each cohort is a MSMP which is independent of all other cohorts. Thus

$$V[Y_j(t)] = \sum_{u=0}^{t} n(t-u)\ P_{1j}(u)(1-P_{ij}(u)), \qquad (28)$$

and

$$COV[Y_i(t),Y_j(t)] = -\sum_{u=0}^{t} n(t-u)\ P_{1i}(u)\ P_{1j}(u). \quad (29)$$

Forming the covariance matrix $V[Y(t)]$ from (28) and (29) it follows that

$$V[S(t)] = V[X(\infty)] + V[Y(t)]. \qquad (30)$$

To measure the effect of policy changes in the transient case, we must first translate these changes into a new $Q(t)$. Then, using (24), we can estimate $P(t)$. If policy changes

and an increase in overall size are to be effected simultaneously, we can consider the entire system to be non-stationary and revise (26) to

$$E[S(t)] = E[W(t)] + E[Y(t)]$$

where

$$E[W_j(t)] = \sum_{i=1}^{6} NP_i \; P_{ij}(t), \quad j = 1,2,\ldots,6.$$

Similarly, $P_{ij}(t)$ replaces P_j in all calculations for $V[X(t)]$ which is subsequently used in a revised (30).

The next case to be considered is a decrease in overall size, initiated by a decrease in the number of new officers entering rank 1; that is, the number of officers leaving the system in a time period is greater than the number of replacements. This may be viewed as merely a diminished recruiting input to gradually reduce size, or as forced attrition in the higher ranks with a revised $Q(t)$ that causes more "departures" to state 1, where "replacement" is no longer on an instantaneous one-to-one basis. A major force reduction would undoubetdly be a combination of both policies.

Let the size of the officer corps prior to time zero be N, and assume steady state conditions. Let $N' < N$ be the desired new size. Then a system of size $(N-N')$ must be emptied to reach this objective size. Figure 4 depicts this process.

At time zero we establish a subsystem of size N', whose members are chosen randomly such that the proportions of officers in each rank are equal to those of the larger size

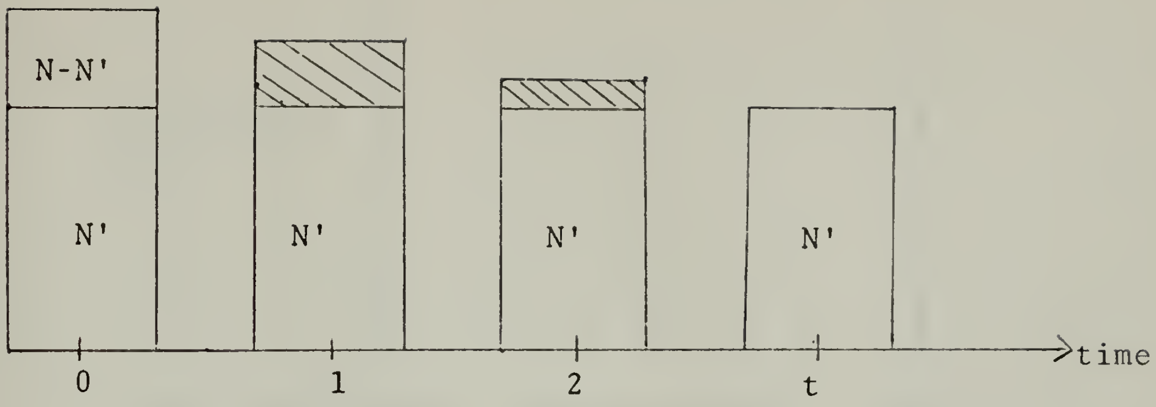

Figure 4. Size Reduction.

N system. These N' officers will proceed in the previously defined closed system model - in either the steady state or transient mode, as appropriate. However, the (N-N') other officers are placed in an open system, where state 0 is defined as "out of the service." Distinguishing the open system by *, we have then $Q^*(t)$ for this open portion as:

$$Q^*(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{10}(t) & 0 & Q_{12}(t) & 0 & 0 & 0 & 0 \\ Q_{20}(t) & 0 & 0 & Q_{23}(t) & 0 & 0 & 0 \\ Q_{30}(t) & 0 & 0 & 0 & Q_{34}(t) & 0 & 0 \\ Q_{40}(t) & 0 & 0 & 0 & 0 & Q_{45}(t) & 0 \\ Q_{50}(t) & 0 & 0 & 0 & 0 & 0 & Q_{56}(t) \\ Q_{60}(t) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For this sub-system, $H_0^*(t) = \sum_j Q_{0j}^*(t) = 0$. Thus, $\overline{H}_0^*(t) = 1$ and using the approximation (24) we can estimate $P^*(t)$. As $t \to \infty$ the limiting form of $P^*(t)$ is given by

$$\lim_{t\to\infty} P^*(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Thus, the expected remaining number in rank j, $E[R_j(t)]$, in the open system can be calculated using the transient part (states 1 through 6) of $P^*(t)$; i.e.,

$$E[R_j(t)] = \sum_{i=1}^{6} (N-N') \, P_i \, P^*_{ij}(t) .$$

As $P^*_{ij}(t) \to 0$ rank j empties to zero. Thus the expected number in the open subsystem, shown as the shaded area in Figure 4, decreases to zero and the total objective size N' is reached. Considering the system as a whole, we have

$$E[S_j(t)] = \frac{N'}{N} \, E[W_j(t)] + E[R_j(t)]; \; j = 1,2,\ldots,6.$$

To conclude this section, we note that a logical next step in the utilization of this model is the construction of $Q(t)$ from actual policy and data. For example, consider a simple development of $Q_{1j}(t)$. Let

α = attrition rate from rank 1 due to death, disability, dismissal, etc.,

β = opportunity for promotion.

T = time in grade 1 until promotion attempt

$\mathrm{EOS} > T$.

Then,

$$Q_{11}(t) = Q^*_{10}(t) = 1-(1-\alpha)^t \qquad t<T$$

$$= 1-(1-\alpha)^T+(1-\beta)(1-\alpha)^T[1-(1-\alpha)^{t-T}] \quad t<EOS$$

$$= 1-(1-\alpha)^T\beta \qquad t\geq EOS$$

$$Q_{12}(t) = Q^*_{12}(t) = 0 \qquad t<T$$

$$= \beta(1-\alpha)^T \qquad t\geq T$$

Q(t) may then be substituted into (24) to obtain an estimate of *P*(t). Analysis of estimation errors involved and the number of convolutions necessary to achieve the desired accuracy can then be conducted.

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1. ORIGINATING ACTIVITY *(Corporate author)* Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE
A SEMI-MARKOV MODEL FOR MILITARY OFFICER PERSONNEL SYSTEMS

4. DESCRIPTIVE NOTES *(Type of report and, inclusive dates)*
Master's Thesis; September 1971

5. AUTHOR(S) *(First name, middle initial, last name)*
Randall W. Austin

6. REPORT DATE September 1971	7a. TOTAL NO. OF PAGES 39	7b. NO. OF REFS 8
8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	

10. DISTRIBUTION STATEMENT
Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

The military officer personnel system is modeled as a semi-Markov process. The probability distributions describing state transitions are developed in terms of policy variables. Effects of changes in promotion policy and force size are studied for both a closed, steady state system and an open, transient situation. Moments of the joint distribution of officers in each rank are presented for all cases. A method for approximation of transient results from known distributions is formulated.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Markov Renewal Process						
Semi-Markov Process						
Manpower Planning						
Military Personnel						
Officer Promotions						
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